WASHINGTON SQUARE COLLEGE OF ARTS AND SCIENCE

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PROPAGATION CONSTANTS OF TRAVELING WAVES

by

MORTIMER L. WEITZ

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ABSTRACT

In this paper we prove rigorously two assumptions which are frequently made in the literature on traveling wave tubes. The first is that in the case of circular cylindrical symmetry involving perfect conductors and dielectrics and a uniform d.c. velocity beam, an amplifying wave must travel more slowly than the electrons. The second assumption we prove is that symmetrical increasing waves cannot exist in the following two special cases: a) a perfectly conducting circular cylinder through which passes an electron beam concentric with the axis of the cylinder and having uniform d.c. velocity and current density, and b) an electron beam with uniform d.c. velocity and current density moving in free space.

I. Introduction

The usual technique employed in the analysis of a structure for possible use as a traveling-wave tube amplifier reduces to the problem of solving an equation the roots of which are the propagation constants of the possible modes of the system. Then one seeks to determine whether any of the roots correspond to an exponentially increasing wave, that is, to an amplifying wave. In general this propagation equation may be transcendental and rather difficult to solve. It would therefore be advantageous to have rules which would limit the portion of the complex plane which has to be examined for possible roots. One might expect that the law of conservation of energy would impose such a limitation. Pierce and Field[1] give a physical argument for believing that such a restriction does indeed exist, but their argument is not proven rigorously. In the present paper, since we wish to make extensive use of a restriction of this type, we shall give a mathematically rigorous proof. We show that a restriction exists for the case of circular cylindrical symmetry involving perfect conductors and dielectrics and a uniform (d.c.) velocity beam; this restriction is in fact that the propagation constant of an amplifying wave must be such that the wave travels more slowly than the (d.c.) electron velocity. (Note that nothing is asserted about the unperturbed wave speed, that is, the speed of propagation of a traveling wave in the absence of the electron beam.)

We then apply this result to show rigorously that symmetrical increasing waves cannot exist in two special cases: 1) a perfectly conducting circular cylinder through which passes an electron beam concentric with the axis of the cylinder and having uniform d.c. velocity and current density, and 2) an electron beam with uniform d.c. velocity and current density moving in free space. (This latter case may be regarded as a special instance of the former, namely, the case where the radius of the cylinder increases to infinity.) These results have been accepted heretofore mainly on the basis of laboratory experiment (which is necessarily limited to a finite number of parameter combinations), but as far as we know they have never been rigorously established.

II. Restriction on the Propagation Constant

Let us consider a finite set of concentric cylindrical shells immersed in a vacuum. Any shell is either a perfect conductor or a perfect homogeneous dielectric and extends to infinity longitudinally. Electrons flow in the positive z-direction (taken to coincide with the axis of the cylinders) between the shells, inside the innermost one and outside the outermost one. The d.c. velocity v_o is uniform, while the electron density is independent of the polar angular coordinate θ and vanishes for sufficient large values of the radial coordinate r. The electromagnetic field components are all assumed to be of the form

$$(1) \qquad f(r)e^{i\omega t - \gamma z}$$

(where γ is a constant), and small-signal theory is assumed applicable. We now show that if γ lies in the second quadrant, that is,

$$(2) \qquad \mathrm{Re}\,\gamma < 0, \ \mathrm{Im}\,\gamma > 0 \ ,$$

then the phase velocity of the wave $\omega/\mathrm{Im}\gamma$ must always be less than the electron velocity v_o. Thus

$$(3) \qquad p_o \equiv \frac{\omega}{v_o} < \mathrm{Im}\gamma .$$

First we show that for any imaginary cylinder of finite length concentric with the z-axis and of sufficiently large radius, the net Poynting vector flux† is outward.

It is well known that in free space the longitudinal field components E_z and H_z satisfy the homogenous wave equation. For fields of the form (1) this equation reduces to

$$(4) \qquad \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) + (\gamma^2 + k^2)f = 0 \ ,$$

which has as solutions the modified Bessel functions $I_o(pr)$ and $K_o(pr)$, where

$$(5) \qquad p^2 = -(\gamma^2 + k^2),$$

$$(6) \qquad k^2 = \omega^2 u_o \epsilon_o .$$

(In terms of the more familiar Bessel and Hankel functions we have

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† cf. reference [3], pp. 131-137.

(7) $$I_n(x) = i^{-n} J_n(ix)$$

$$K_n(x) = \frac{\pi}{2} i^{n+1} H_n^{(1)}(ix) .)$$

Since (4) involves p^2 rather than p we may restrict our consideration to values of p lying in the first and fourth quadrants. In the case of many-valued functions like K_0, we consider only the principal branch. Thus

(8) $$-\frac{\pi}{2} < \text{ph } p < \frac{\pi}{2} .$$

Then it follows from the asymptotic formulas

(9) $$I_n(x) \sim \frac{e^x}{\sqrt{2\pi x}}$$

and

$$K_n(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left[1 + \frac{4n^2-1}{8x}\right]$$

that the only linear combinations

(10) $$AK_0(pr) + BI_0(pr)$$

which do not grow exponentially as $r \to \infty$ are those in which

(11) $$B = 0.$$

Hence we conclude that in our problem in the charge-free region beyond the largest physical cylinder and the outermost electrons, the E_z and H_z fields are of the form

(12) $$K_0(pr)\, e^{i\omega t - \gamma z}.$$

Also, from Maxwell's equations it follows that the H_θ and E_θ fields are of the form

$$K_1(pr)\, e^{i\omega t - \gamma z} .$$

Hence in view of (9) it follows that the Poynting vector flux through the curved surface of our fictitious cylinder tends to zero as the radius of the cylinder increases to infinity; that is,

(13) $$\iint_S (\vec{E} \times \vec{H}^*)_n \, dS \to 0$$

as $r \to \infty$. Here the asterisk indicates complex conjugate.

Now we must consider the flux that passes outward through the end faces. Here the expression for the flux is

$$(14) \qquad \pm \iint_S (E_r H_\theta^* - E_\theta H_r^*)\, dS.$$

(The minus sign is used for the face with the smaller value of z.) Clearly the portions of the surface of integration S which are intersected by perfectly conducting cylinders do not contribute. The rest of the space is occupied by perfect dielectrics in which the following relations hold:

$$(15) \qquad E_r = \frac{\gamma}{i\omega\varepsilon} H_\theta$$

$$(16) \qquad E_\theta = -\frac{i\omega\mu}{\gamma} H_r .$$

(15) and (16) follow from the homogeneous Maxwell equations. (In the portion of the space through which the electrons flow, ε and μ become ε_o and μ_o respectively.) Inserting (15) and (16) in (14) we obtain

$$(17) \qquad \pm \iint_S \left(\frac{\gamma}{i\omega\varepsilon} |H_\theta|^2 + \frac{i\omega\mu}{\gamma} |H_r|^2\right) dS.$$

We see from (2) that the integral (with the positive sign) must have a positive real part. Since this real part increases monotonically with r and z, it follows from (13) that for cylinders of sufficiently large radius the net flux is outward.

Using Poynting's theorem we find that for any of these sufficiently large cylinders,

$$(18) \qquad \mathrm{Re} \iiint_V E_z J_z^* \, dv < 0 ,$$

where J_z is the a.c. current density, and V is the volume of the cylinder. Now

$$(19) \qquad E_z J_z^* = \frac{E_z}{J_z} J_z J_z^* .$$

Hence (18) implies that at least for some value of r,

$$(20) \qquad \mathrm{Re} \frac{E_z}{J_z} < 0 .$$

The electronic impedance E_z/J_z has the form (see e.g. [2] , eqn. (18))

$$(21) \qquad E_z / J_z = \frac{v_o^3 (i\, p_o - \gamma)^2}{-i\omega e/mJ_o} ,$$

where J_0 is the d.c. current density in the positive z-direction and hence is negative.

Thus (20) implies that

$$(22) \qquad \mathrm{Re}\, \frac{(i\, p_o - \gamma)^2}{i} < 0 ;$$

that is, $\gamma - i\, p_o$ lies in the second or fourth quadrant. Since the latter alternative is ruled out by (2), it follows that (3) holds.

One obvious generalization is that if v_o varies with z and r but always remains positive, then for some value of r for every z, equation (20) and hence (3) must hold. In other words in every transverse section there must be some circle on which the electron velocity is greater than the phase velocity of the traveling wave.

If we wish to include the case of imperfect dielectrics in this generalization, we must replace (15) by

$$(23) \qquad E_r = \frac{\gamma}{\sigma + i\omega\varepsilon} H_\theta ,$$

where σ is the conductivity. But then it no longer follows that the integral appearing in (14) must have a positive real part. However, if the dielectrics are not isotropic, then conduction in the z and θ directions could be permitted, if the radial conductivity remains zero. Then (15) would still hold and the rest of the argument proceeds as above. In the case of an infinitesimally thin shell there is no radial current flow (nor, for that matter, any finite contribution to (14)), and so here too our results apply. Thus these results may be applied to the usual helical sheath idealization (cf. [5] App. B or [2], p. 2).

III. Application to a General Uniform Beam.

We now consider the special case where the electron flow is confined to a single beam with uniform (d.c.) current density as well as uniform velocity. That is, J_0 has a constant negative value from $r = 0$ to $r = b$, say, and is zero for $r > b$, while the cylindrical shells (if any) lie outside of b. Within the beam, E_z satisfies the inhomogeneous wave equation

$$(24) \quad \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial E_z}{\partial r}\right) + (\gamma^2 + k^2)\, E_z = \frac{-(\gamma^2 + k^2)}{i\omega\varepsilon_o} J_z ,^{\dagger}$$

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† Cf. [2], eq. (4).

or, in view of (21),

$$(25)\quad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z}{\partial r}\right) + (\gamma^2 + k^2)\left[1 - \frac{e/m\ J_0}{\varepsilon_0 v_0^3 (i\ p_0 - \gamma)^2}\right] E_z = 0.$$

Thus f(r), the radial part of E_z, must satisfy

$$(26)\quad \frac{1}{r}\frac{d}{dr}\left(r\frac{df}{dr}\right) - \eta^2 f = 0,$$

where

$$(27)\quad \eta^2 = p^2\left[1 - \frac{e/m\ J_0}{\varepsilon_0 v_0^3 (i\ p_0 - \gamma)^2}\right].$$

Hence

$$(28)\quad f(r) = A\ I_0(\eta r) + B\ K_0(\eta r).$$

Since K_0 is unbounded at the origin, we must set B = 0. Then we have

$$(29)\quad E_z = A\ I_0(\eta r)\ e^{i\omega t - \gamma z}.$$

For later use we wish to derive a relation concerning the radial logarithmic derivative of (29) evaluated at the edge of the beam

$$(30)\quad \frac{1}{E_z}\frac{\partial E_z}{\partial r}\bigg|_{r=b} = \eta\frac{I_1(\eta b)}{I_0(\eta b)}.$$

Specifically we want to show that for values of γ satisfying (2) (and hence (3)), and for

$$(31)\quad J_0 < 0,$$

the following inequality holds:

$$(32)\quad I_m\,\eta\frac{I_1(\eta b)}{I_0(\eta b)} > 0.$$

We first establish that under the above conditions,

$$(33)\quad I_m\,\eta^2 > 0.$$

We may rewrite (27) as follows:

$$(34) \quad \eta^2 = p^2 + \frac{Kp^2}{(\gamma - i\,p_o)^2} = p^2 - K\,\frac{\gamma + ik}{\gamma - i\,p_o} \cdot \frac{\gamma - ik}{\gamma - i\,p_o},$$

where K is a positive constant:

$$(35) \quad K \equiv \frac{-e/m\ J_o}{\varepsilon_o v_o^3}.$$

In (34) the first term on the right, p^2, has a positive imaginary part (from (2)). That the second term also has a positive imaginary part may be seen by noting that as a result of (3) and the physical requirement

$$(36) \quad k < p_o,$$

both factors

$$\frac{\gamma + ik}{\gamma - i\,p_o} \quad \text{and} \quad \frac{\gamma - ik}{\gamma - i\,p_o}$$

lie in the fourth quadrant. Hence ther product must lie below the real axis, and (33) follows immediately.

We now complete the demonstration of (32). Let

$$(37) \quad \xi = \eta\, b;$$

then we must show that

$$(38) \quad \mathrm{Im}\ \xi\,\frac{I_1(\xi)}{I_0(\xi)} > 0.$$

Now

$$(39) \quad \frac{\xi I_1(\xi)}{I_o(\xi)} = \xi\,\frac{d}{d\xi}\,\log I_0(\xi),$$

or, using the product representation

$$(40) \quad I_0(\xi) = \prod_{n=1}^{\infty}\left(1 + \frac{\xi^2}{g_n^2}\right)^{\dagger}$$

(where $\pm\, ig_n$ are the pure imaginary zeroes of I_0),

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† cf. [4], p. 498.

$$(41) \qquad \frac{\xi I_1(\xi)}{I_0(\xi)} = 2\sum_{n=1}^{\infty} \frac{\xi^2}{\xi^2 + g_n^2} \quad .$$

From (33) it follows that each term of the series lies above the real axis. Hence also their sum must also lie above the real axis and (38) and (32) follow immediately.

IV. Uniform Beam Within a Cylinder

The usefulness of this last result can be illustrated by some particular cases. First we suppose the uniform beam is concentrically surrounded by a perfectly conducting cylinder. We wish to show that in this case the γ equation will not have any roots satisfying (2).

We first show that the γ equation has the form

$$(42) \qquad \frac{\eta I_1(\eta b)}{I_0(\eta b)} = \frac{p\, I_1(pb) + K_1(pb)\, I_0(pa)/K_0(pa)}{I_0(pb) - K_0(pb)\, I_0(pa)/K_0(pa)} \quad ,$$

where a is the radius of the cylinder.

In the charge-free space between the beam and the cylinder, the E_z field must have the form

$$(43) \qquad E_z = \left[I_0(pr) - \frac{I_0(pa)}{K_0(pa)} K_0(pr) \right] e^{i\omega t - \gamma z} \quad ,$$

since it must vanish at $r = a$. We note that the right side of (42) is merely the limit as $r \to b^+$ of the radial logarithmic derivative of this field, and from (30) we see that the left side of (42) is the limit as $r \to b^-$. Thus (42) must be true if $\frac{\partial E_z}{\partial r}$ is continuous as well as E_z. But from Maxwell's equation

$$(44) \qquad \frac{\partial E_z}{\partial r} = i\omega\mu_0 H_\theta - \gamma E_r$$

and from the absence of surface charge and current at the edge of the beam, we know that H_θ, E_r, and hence $\frac{\partial E_z}{\partial r}$, must be continuous.

To demonstrate the non-existence of roots satisfying (2) we need merely show, in view of equation (32), that the imaginary part of the right side of (42) is always non-positive. (Now p must satisfy

$$(45) \qquad \left.\begin{array}{l} \mathrm{Re}\ p > 0 \\ \mathrm{Im}\ p > 0 \end{array}\right\}$$

as a consequence of (2) and (8).)

As indicated above, the right side of (42) is merely

$$(46) \qquad \frac{1}{w} \frac{dw}{dr}\bigg|_{r=b} ,$$

where w satisfies

$$(47) \qquad \frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} - p^2 w = 0;$$

$$(48) \qquad w = 0 \quad \text{at } r = a.$$

Setting

$$(49) \qquad v(r) = \frac{1}{w}\frac{dw}{dr} ,$$

we obtain the Riccati equation

$$(50) \qquad \frac{dv}{dr} = p^2 - v^2 - \frac{v}{r} ;$$

$$(51) \qquad v = \infty \quad \text{at } r = a.$$

We change the independent variable, setting

$$(52) \qquad R = a - r ,$$

and obtain

$$(53) \qquad \frac{dv}{dR} = v^2 + \frac{v}{a-R} - p^2 ;$$

$$v = \infty \quad \text{at } R = 0.$$

The power series solution is of the form

$$(54) \qquad v = -\frac{1}{R} - \frac{1}{2a} - \left(\frac{5}{12a^2} + \frac{p^2}{3}\right) R + \cdots ,$$

so for sufficiently small R,

$$(55) \qquad \mathrm{Im}\, v < 0$$

in view of the fact that

$$(56) \qquad \mathrm{Im}(-p^2) < 0.$$

It is now clear that

$$(57) \qquad \mathrm{Im}\, v = 0, \quad \mathrm{Re}\ v < \infty$$

cannot occur even as R increases since at such a point

$$(58) \qquad \frac{d}{dR}(\mathrm{Im}\, v) \geq 0,$$

which contradicts (53).

Furthermore if for some value of R (R_1 say) between 0 and a, v should go to infinity, then an expansion of v around $R = R_1$ would yield a series similar to (54) and as above it would follow that for sufficiently small and positive values of $R - R_1$, $\mathrm{Im}\, v < 0$. Since v(r) does not have any jump discontinuities, it follows that

$$(59) \qquad \mathrm{Im}\, v \leq 0 \quad .$$

V. Uniform Beam in Free Space.

Next we show that also in the particular case of a uniform beam in free space the γ equation will not have roots satisfying (2). The γ equation for this case is

$$(60) \qquad \eta \frac{I_1(\eta b)}{I_0(\eta b)} = - p \frac{K_1(pb)}{K_0(pb)} .$$

As in the preceding section, our problem reduces to showing that when p satisfies (45) the right side of the γ equation does not have a positive imaginary part. Thus we wish to demonstrate that

$$(61) \qquad \mathrm{Im}\left\{- p \frac{K_1(pb)}{K_0(pb)}\right\} \leq 0 \ .$$

Again the expression to be investigated may be regarded as a logarithmic derivative. It satisfies the same Riccati equation (50) as above. However (51) no longer holds. Instead we observe from (9) that when p satisfies (45) and r (positive) is sufficiently large,

$$(62) \qquad v(r) \equiv - p \frac{K_1(pr)}{K_0(pr)}$$

must lie below the real axis. Again, using essentially the same argument as above, we find from (50) that for smaller (but still positive) values of r

$$\mathrm{Im}\, v \leq 0 \ ;$$

this completes the demonstration.

We summarize the last two sections as follows: we have shown that in the two cases of a uniform beam in free space and a uniform beam located symmetrically within a perfectly conducting cylinder, it is impossible to have symmetric waves increasing and propagating in the direction of the electron flow.

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